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Reed–Muller-Type Codes Over the Segre Variety

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The a-invariant is determined and a description of the defining ideal for the set S_K of rational points of the Segre variety over a finite field K is given. The dimension as well as the minimum distance of a Reed–Muller-type linear code defined over S_K are also determined. An example is given to illustrate the ideas.

Key Words: Reed–Muller codes; Segre variety; a-invariant; defining ideal.

1. INTRODUCTION

Let $K = GF(q)$ be a finite field with $q = p^r$ elements (p prime and r a positive integer). Let $\mathbf{P}_l(K)$ be the l-projective space over K and let $X = \{P_1, \ldots, P_t\}$ be a subset of $\mathbf{P}_l(K)$. If $\mathscr{L}$ is a finite K-dimensional linear space of functions defined on the set X with values in the field K, the evaluation map

$$ev_X\colon \mathscr{L} \to K^t, \quad ev_X(f) = (f(P_1), \ldots, f(P_t))$$

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determines the K-linear code $C_X = ev_X(\mathscr{L})$. For any subset X and a general linear subspace of functions $\mathscr{L}$, it is a difficult task to determine the parameters of the code C_X.

Let $A = K[X_0, \ldots, X_l] = \oplus_{i \geq 0} A_i$ be the graded polynomial ring over the field K where A_i is the K-subspace generated by all monomials in A of degree i. If $X \subseteq \mathbf{P}_l(K)$ and $\mathscr{L} = A_d$, $(d \geq 1)$ is the d-graded homogeneous component of the ring A, the corresponding linear code $C_X(d) := ev_X(A_d)$, will be called the Reed–Muller linear code over the set X. If $I_X = \{f \in A: f(P) = 0\ \forall P \in X\} = \oplus_{i \geq 0} I_X(i)$ is the (graded) vanishing ideal of the set X, the linear code $C_X(d)$ is isomorphic to the K-vector space $A_d/I_X(d)$. Thus, the dimension of the code is given by the *Hilbert function* of the ring A/I_X. A generating matrix for this code can be obtained by finding a Gröbner basis for the ideal I_X such that the cosets, with respect to this ideal, of the monomials in A of degree d which are not in the leading terms ideal $LT(I_X)$, are a K-basis for the vector space $A_d/I_X(d)$. If $\mathscr{B}$ is this set of monomials then $(ev_X(\mathscr{M}))_{\mathscr{M} \in \mathscr{B}}$ is a generating matrix for the code $C_X(d)$.

Several authors have provided results about the code $C_X(d)$ or related codes for different sets X [2, 4, 8–12]. The Segre embedding has been used recently to prove a conjecture on the weight of product codes [13] proposed by Wei and Yang [15]. The purpose of this paper is to determine the a-invariant, the Hilbert function of the ring A/I_X, a set of generators for the ideal I_X, as well as the minimum distance of the code $C_X(d)$ when X is the set of K-rational points of the Segre variety defined over the finite field K.

2. GENERAL RESULTS

In this section, we recall some basic definitions such as the Hilbert function, the a-invariant of an ideal, the Segre variety and general results that will be useful later on.

Let K, $X \subseteq \mathbf{P}_l(K)$ with $|X| = t$, A and $I = I_X$ be as defined in the previous section and let $R = A/I$ be the coordinate ring of X. The *Hilbert function* of R is defined as $H_X(d) = dim_K A_d - dim_K I_d$, (cf. [1, Chapter 9, Section 3]).

Let $I = \oplus_{i=\gamma}^{\infty} I_i$, where γ is the lowest degree of a nontrivial homogeneous component of the ideal I. The a-invariant of R (or the a-invariant of I or even the a-invariant of X) is the integer a_X such that

(a) $H_X(d) = dim_K A_d = \binom{l+d}{d} \Leftrightarrow d < \gamma$;

(b) $H_X(d) < H_X(d+1) < t$ if $0 \leq d < a_X$;

(c) $H_X(d) = t$ if $d > a_X$.

The number $a_X + 1$ is called the *regularity index* of R by some authors.

For any field L let $\mathbf{P}_r(L)$ be the r-dimensional projective space over the field L. Recall that the Segre map is defined as [6 p. 25, 7, Section 25.5]

$$\varphi\colon \mathbf{P}_n(L) \times \mathbf{P}_m(L) \to \mathbf{P}_N(L),$$
$$\varphi(\underline{a}, \underline{b}) = (a_0 b_0, \ldots, a_i b_j, \ldots, a_n b_m),$$

where $\underline{a} = (a_0, \ldots, a_n)$, $\underline{b} = (b_0, \ldots, b_m)$ and $N = (n+1)(m+1) - 1$.

The image of this map is known as the Segre variety defined over the field L and it will be denoted by S. If $Z_{ij} = X_i Y_j$ this variety is the locus of the quadratic polynomials $Z_{ij}Z_{kl} - Z_{il}Z_{kj}$.

From now on, an element of $\mathbf{P}_N(L)$ will be denoted by $\underline{z} = (z_{00}, \ldots, z_{nm})$ and let $A_n = L[X_0, \ldots, X_n]$, $A_m = L[Y_0, \ldots, Y_m]$ and $A_N = L[Z_{00}, \ldots, Z_{nm}]$.

In the sequel, L will denote the finite field K with $q = p^r$ elements (p prime and r a positive integer) and $S = S_K$ the set of K-rational points of the Segre variety defined over K.

Note that, since the Segre map φ is an embedding,

$$\#(S_K) = \#(\mathbf{P}_n(K))\#(\mathbf{P}_m(K)) = \left(\frac{q^{n+1}-1}{q-1}\right)\left(\frac{q^{m+1}-1}{q-1}\right).$$

Let $\bar{K}$ be the algebraic closure of K and $\bar{X}$ be the image of the map $\bar{\varphi}\colon \mathbf{P}_n(\bar{K}) \times \mathbf{P}_m(\bar{K}) \to \mathbf{P}_N(\bar{K})$, given by $\bar{\varphi}(\underline{a}, \underline{b}) = (a_0 b_0, \ldots, a_i b_j, \ldots, a_n b_m)$. If $A_N = K[Z_{00}, \ldots, Z_{nm}]$ let $\bar{A}_N = \bar{K}[Z_{00}, \ldots, Z_{nm}]$ and let $I_{\bar{X}}$ be the vanishing ideal of $\bar{X}$ in $\bar{A}_N$.

Remark 2.1. Since $\bar{K}$ is algebraically closed $I_{\bar{X}} = \langle I_{\bar{X}}(2) \rangle$, (cf. [6, p. 51]).

Remark 2.2. The following result, which will be useful later, was proved in [9] (see also [10]): $I_{\mathbf{P}_n(K)} = \langle X_i^q X_j - X_i X_j^q : 0 \leq i < j \leq n \rangle$.

3. THE HILBERT FUNCTION OF S_K

Let K be a finite field with q elements and let $S = S_K$ be the set of K-rational points of the Segre variety as introduced above. For any $I = (i_0, \ldots, i_n) \in \mathbf{N}^{n+1}$, X^I will denote the monomial $X_0^{i_0} \cdots X_n^{i_n}$. Let $B = K[X_0 Y_0, \ldots, X_n Y_m] = \oplus_{d \geq 0} B_d$ be the graded subalgebra of $K[X_0, \ldots, X_n, Y_0, \ldots, Y_m]$ with the grading given by

$$B_d = \left\{ \sum_{I,J} a_{I,J} X^I Y^J : a_{I,J} \in K, |I| = |J| = d \right\}, \quad d \geq 0.$$

LEMMA 3.1. *For $d \geq 2$ the kernel of the surjective linear transformation $\theta{:}A_N(d) \to B_d$, $f \to \theta_f$, where $\theta_f(\underline{X}, \underline{Y}) = f(X_0Y_0, \ldots, X_nY_m)$, is $I_S(2)A_N(d-2)$.*

Proof. It is easy to see that $I_S(2)A_N(d-2) \subset ker\,\theta$. Obviously, $ker\,\theta \subset I_{\bar{S}} = \langle Q_{ij} \rangle$ with $Q_{ij} \in A_N$ being quadrics. By using the generalized division algorithm, any $f \in ker\,\theta$ can be written as $f = \sum_{i,j} \lambda_{ij}Q_{ij} + r$, where $\lambda_{ij} \in A_N(d-2)$ and $r \in A_N$ is not divisible by $LT(Q_{ij})$, the leading term of Q_{ij}. Since $\{Q_{ij}\}$ forms a Gröbner basis for $I_{\bar{S}}$ it follows that $r = 0$ (since $f \in I_{\bar{S}}$) and the claim is proved. ■

Let $V_d = I_{\mathbf{P}_n}(d)A_m(d) + A_n(d)I_{\mathbf{P}_m}(d) \subset B_d$. The following result will be useful in determining the Hilbert function of the ring A_N/I_S.

PROPOSITION 3.1. *For $d \geq 2$ the kernel of the surjective linear transformation $\pi \circ \theta$ given by*

$$A_N(d) \xrightarrow{\theta} B_d \xrightarrow{\pi} B_d/V_d$$

$$f \to \theta_f \to \theta_f + V_d$$

is $I_S(d)$.

Proof. If $f \in ker(\pi \circ \theta)$ then $\theta_f \in V_d$. In this case $\theta_f = \sum_i (h_im_i(Y) + M_i(X)h_i')$ where $h_i \in I_{\mathbf{P}_n}(d)$, $m_i(Y) \in A_m(d)$, $M_i(X) \in A_n(d)$ and $h_i' \in I_{\mathbf{P}_m}(d)$. Therefore $f(\varphi(P,Q)) = \theta_f(P,Q) = 0$ $\forall P \in \mathbf{P}_n(K)$, $Q \in \mathbf{P}_m(K)$, and it follows that $f \in I_S(d)$.

If $f \in I_S(d)$ then $\theta_f(P,Q) = 0$ $\forall P \in \mathbf{P}_n(K)$, $Q \in \mathbf{P}_m(K)$, i.e., $\theta_f \in I_{\mathbf{P}_s(K)}$ with $s = m + n + 1$. The following cases arise:

(1) $d < q + 1$, $2d < q + 1$. Using Remark 2.2 above and the fact that $\theta_f \in B_d$, it follows that $\theta_f \equiv 0$ and therefore $f \in V_d$.

(2) $d < q + 1$, $2d \geq q + 1$. Since $\theta_f \in B_d$ and B_d consists of bihomogeneous polynomials of bidegree (d,d), it follows that $\theta_f \equiv 0$ and therefore $f \in V_d$.

(3) $d \geq q + 1$. In this case,

$$\theta_f = \sum_u m_u(\underline{X}, \underline{Y})(X_i^qX_j - X_iX_j^q) + \sum_v M_v(\underline{X}, \underline{Y})(Y_i^qY_j - Y_iY_j^q)$$

where $m_u(\underline{X}, \underline{Y})$ and $M_v(\underline{X}, \underline{Y})$ are monomials of degree $2d - (q+1)$. However,

$$\theta_f = \sum_u m_u'(\underline{X})(X_i^qX_j - X_iX_j^q)m_u''(\underline{Y}) + \sum_v M_v'(\underline{Y})(Y_i^qY_j - Y_iY_j^q)M_v''(\underline{X})$$

with $m_u''(\underline{Y}) \in A_m(d)$ and $M_v''(\underline{X}) \in A_n(d)$. Thus $\theta_f \in V_d$. ■

COROLLARY 3.1. *The Hilbert function of the ring A_N/I_S is given by*

$$H_S(d) = H_{\mathbf{P}_n(K)}(d)H_{\mathbf{P}_m(K)}(d).$$

Proof. From Proposition 3.1 it follows that $A_N(d)/I_S(d) \simeq B_d/V_d$. Moreover, $B_d/V_d \simeq A_n(d)/I_{\mathbf{P}_n(K)}(d) \otimes_K A_m(d)/I_{\mathbf{P}_m(K)}(d)$, (the natural isomorphism). By taking dimensions the claim follows. ■

Remark 3.1. From the above corollary it follows that

$$dim_K\, C_S(d) = H_{\mathbf{P}_n(K)}(d)H_{\mathbf{P}_m(K)}(d).$$

COROLLARY 3.2. *The a-invariant of the ring A_N/I_S is given by*

$$a_S = \max\{n(q-1), m(q-1)\}.$$

Proof. It is an immediate consequence of the previous corollary and the fact that $a_{\mathbf{P}_n(K)} = n(q-1)$ and $a_{\mathbf{P}_m(K)} = m(q-1)$ (cf. [10]). ■

4. THE VANISHING IDEAL OF S_K

The following result provides a set of generators for the vanishing ideal of the set S of K-rational points of the Segre variety defined over the finite field K with q elements.

THEOREM 4.1. *The vanishing ideal of the set S of* K*-rational points of the Segre variety is given by*

$$I_S = \langle I_S(2), I_S(q+1)\rangle.$$

Proof. In the proof of Proposition 3.1 it was noted that if $f \in I_S(d)$ for $d < q+1$, then $\theta_f \equiv 0$. Therefore, $f \in I_S(2)A_N(d-2)$, i.e., $I_S(d) \subset \langle I_S(2)\rangle$ whenever $d < q+1$.

If $d > q+1$, in order to prove the theorem, it is enough to show that $I_S(d) \subset \sum_{i,j} Z_{ij}I_S(d-1)$. In this case, with the notation as in Proposition 3.1, we observe that if $m_u(\underline{X}, \underline{Y}) = X_0^{i_0}\cdots X_n^{i_n}Y_0^{j_0}\cdots Y_m^{j_m}$ and $M_v(\underline{X}, \underline{Y}) = X_0^{a_0}\cdots X_n^{a_n}Y_0^{b_0}\cdots Y_m^{b_m}$, then $\sum_{r=0}^n i_r > 0$, $\sum_{r=0}^m j_r > 0$, $\sum_{r=0}^n a_r > 0$ and $\sum_{r=0}^m b_r > 0$. Therefore, $\theta_f = \sum_{i,j} X_iY_jh_{ij}(\underline{X}, \underline{Y})$ with $h_{ij}(\underline{X}, \underline{Y}) \in I_{\mathbf{P}_s(K)}(2d-2)$. Thus $f = \sum_{i,j} Z_{ij}\theta_{ij}(\underline{Z})$ with $\theta_{ij}(\underline{Z}) \in I_S(d-1)$, showing that $f \in \sum_{i,j} Z_{ij}I_S(d-1)$. ■

5. THE MINIMUM DISTANCE OF THE CODE $C_S(d)$

Let K be a finite field with q elements and let $S = S_K$ be the set of K-rational points of the Segre variety. In this section the minimum distance of the Reed–Muller-type code $C_S(d)$ is determined.

Recall that the minimum distance of the projective Reed–Muller code $C_{\mathbf{P}_l(K)}(d)$ is $\delta_{\mathbf{P}_l}(d) = (q-s)q^{l-r-1}$, where $d-1 = r(q-1)+s,\ 0 \leq s < q-1$ (cf. [14]).

THEOREM 5.1. *With the same notation as above, the minimum distance $\delta_S(d)$ of the code $C_S(d)$ is given by*

$$\delta_S(d) = \delta_{\mathbf{P}_n}(d)\delta_{\mathbf{P}_m}(d).$$

Proof. Let $f \in K\,[Z_{00}, \ldots, Z_{nm}]_d$ and let $S = \{P_{11}, \ldots, P_{k_1k_2}\}$ where $k_1 = |\mathbf{P}_n(K)|$ and $k_2 = |\mathbf{P}_m(K)|$. Let $\Lambda = (f(P_{11}), \ldots, f(P_{k_1k_2})) \in C_S(d) - \{0\}$. Note that if φ is the Segre map, then for all i, j, $P_i \in \mathbf{P}_n(K)$ and $Q_j \in \mathbf{P}_m(K)$ exist such that $\varphi(P_i, Q_j) = P_{ij}$, i.e., $(X_0Y_0, \ldots, X_nY_m)(P_r, Q_s) = P_{rs}$. Therefore,

$$\Lambda = (f(X_0Y_0, \ldots, X_nY_m)(P_1, Q_1), \ldots, f(X_0Y_0, \ldots, X_nY_m)(P_{k_1}, Q_{k_2})).$$

Let $f(X_0Y_0, \ldots, X_nY_m) = \sum_{i,j} a_{I,J}X^IY^J$. For each $P \in \mathbf{P}_n(K)$ and $Q \in \mathbf{P}_m(K)$ let $f_P(Y) = \sum a_{I,J}P^IY^J \in A_m(d)$ and $f_Q(X) = \sum a_{I,J}X^IQ^J \in A_n(d)$, respectively.

Let $\Lambda_i = (f_{P_i}(Q_1), \ldots, f_{P_i}(Q_{k_2})) \in C_{\mathbf{P}_m}(d)$, $i = 1, \ldots, k_1$, and let $k_3 = \#\{i : \Lambda_i \neq 0\} > 0$. Since the Hamming weight of Λ_i is such that $wt(\Lambda_i) \geq \delta_{\mathbf{P}_m}(d)$ for all i with $\Lambda_i \neq 0$, it follows that $wt(\Lambda) \geq k_3\delta_{\mathbf{P}_m}(d)$.

In a similar way, for each $j \in \{1, \ldots, k_2\}$ let $\Gamma_j = (f_{Q_j}(P_1), \ldots, f_{Q_j}(P_{k_1})) \in C_{\mathbf{P}_n}(d)$. Take j such that $\Gamma_j \neq 0$. If $k_3 < \delta_{\mathbf{P}_n}(d)$ then $wt(\Gamma_j) \leq k_3 < \delta_{\mathbf{P}_n}(d)$. Thus $wt(\Lambda) \geq \delta_{\mathbf{P}_n}(d)\delta_{\mathbf{P}_m}(d)$.

Let $\Omega_1 = (g(P_1), \ldots, g(P_{k_1})) \in C_{\mathbf{P}_n}(d)$ with $wt(\Omega_1) = \delta_{\mathbf{P}_n}(d)$ and $g \in A_n(d)$, and similarly let $\Omega_2 = (h(Q_1), \ldots, h(Q_{k_2})) \in C_{\mathbf{P}_m}(d)$ with $wt(\Omega_2) = \delta_{\mathbf{P}_m}(d)$ and $h \in A_m(d)$. Since the linear transformation θ (as defined in Proposition 3.1) is surjective, $F \in A_N(d)$ exists such that

$$(gh(P_1, Q_1), \ldots, gh(P_{k_1}, Q_{k_2})) = (F(\varphi(P_1, Q_1)), \ldots, F(\varphi(P_{k_1}, Q_{k_2}))) =$$

$\Omega \in C_S(d)$ and $wt(\Omega) = \delta_{\mathbf{P}_n}(d)\delta_{\mathbf{P}_m}(d)$, proving the Theorem. ∎

6. AN EXAMPLE

In order to illustrate the main results given above, an example is provided.

Let $K = GF(4) = \{0, 1, a, a^2\}$ be the field with four elements and let φ: $\mathbf{P}_1(K) \times \mathbf{P}_1(K) \to \mathbf{P}_3(K)$ be the Segre embedding over K. It is easy to see that the set of K-rational points of the Segre variety is the following:

$$S = \{(0,0,0,1),(0,0,1,0),(0,1,0,0),(1,0,0,0),(0,0,1,a),(0,0,1,a^2),$$

$$(0,0,1,1),(0,1,0,a),(1,a,a,a^2),(1,0,a,0),(1,a^2,a,1),(1,1,a,a),$$

$$(1,a,0,0),(1,a^2,0,0),(1,1,0,0),(0,1,0,a^2),(1,a,a^2,1),(1,0,a^2,0),$$

$$(1,a^2,a^2,a),(1,1,a^2,a^2),(0,1,0,1),(1,a,1,a),(1,0,1,0),(1,a^2,1,a^2),$$

$$(1,1,1,1)\}.$$

Since $a_{\mathbf{P}_1(K)} = 3$, it follows from Corollary 3.2 that $a_S = 3$. Now take $d = 2$. Then from Corollary 3.1, $dim_K\, C_S(2) = H_S(2) = 9$, from Theorem 5.1 the minimal distance of the code is 9, and from Theorem 4.1, $I_S = \langle I_S(2), I_S(5)\rangle$. If $Z_{00}, Z_{10}, Z_{01}, Z_{11}$ denote the coordinates of $\mathbf{P}_3(K)$, a set of generators for I_S is

$$\{Z_{10}Z_{01} - Z_{00}Z_{11}, Z_{01}^4Z_{11} - Z_{01}Z_{11}^4, Z_{00}Z_{01}^3Z_{11} - Z_{00}Z_{11}^4, Z_{00}^2Z_{01}^2Z_{11}$$

$$- Z_{00}Z_{10}Z_{11}^3, Z_{00}^3Z_{01}Z_{11} - Z_{00}Z_{10}^2Z_{11}^2, Z_{10}^4Z_{11} - Z_{10}Z_{11}^4, Z_{00}Z_{10}^3Z_{11}$$

$$- Z_{00}Z_{11}^4, Z_{00}^2Z_{10}^2Z_{11} - Z_{00}Z_{01}Z_{11}^3, Z_{00}^3Z_{10}Z_{11} - Z_{00}Z_{01}^2Z_{11}^2, Z_{00}^4Z_{11}$$

$$- Z_{00}Z_{11}^4, Z_{00}^4Z_{01} - Z_{00}Z_{01}^4, Z_{00}^4Z_{10} - Z_{00}Z_{10}^4\}.$$

Now, if $Z_{ij} = X_iY_j$ for all $i = 0, 1$, $j = 0, 1$, we observe that the generators for I_S can be obtained from the elements of the K-linear space $V = [I_{\mathbf{P}_1(K)}(2) \otimes_K A_1(2)] + [A_1(2) \otimes_K I_{\mathbf{P}_1(K)}(2)]$.

For instance, if we take $H = Y_0Y_1^4(X_0^4X_1 - X_0X_1^4) \in V$, we note that:

$$H = (X_0Y_0)(X_0^3Y_1^3)(X_1Y_1) - X_0Y_0(X_1Y_1)^4 = Z_{00}Z_{01}^3Z_{11} - Z_{00}Z_{11}^4 \in I_S.$$

Also, a basis $\{f_1, f_2, \ldots, f_9\}$ for the vector space $A_3(2)/I_S(2)$ can be given: $f_1 = Z_{00}^2$, $f_2 = Z_{00}Z_{01}$, $f_3 = Z_{00}Z_{10}$, $f_4 = Z_{00}Z_{11}$, $f_5 = Z_{01}^2$, $f_6 = Z_{01}Z_{11}$, $f_7 = Z_{10}^2$, $f_8 = Z_{10}Z_{11}$, $f_9 = Z_{11}^2$, from which a generating matrix for the code $C_S(2)$ can be obtained. The Macaulay System [3] was used in the calculations of this example.

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